Questions 5. (20 Marks) Consider the time-varying system

$$x' = f(t, x), \tag{TS}$$

where $f(t, x)$ is continuous and locally Lipschitz in x for all $t \geq 0$ and $x \in D \subset R^n$; $f(t, 0) \equiv 0$ for all $t \geq 0$ and $x = 0 \in D$. Let $V : [0, \infty) \times D \to R$ be of C^1 such that

$$W_1(x) \leq V(t, x) \leq W_2(x); \tag{5a}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0; \tag{5b}$$

$$V(t+\delta, \varphi(t+\delta; t, x)) - V(t, x) \leq -\lambda V(t, x), \tag{5c}$$

for all $t \geq 0$, $x \in D$ and for some $\delta > 0$, where $W_j(x) > 0$ ($j = 1, 2$) are positive definite on D, $0 < \lambda < 1$, and $\varphi(s; t, x)$ is the solution of (TS) that starts at (t, x).

1. Show that $x = 0$ is uniformly asymptotically stable.
2. If all the conditions hold globally and $W_1(x)$ is radially unbounded, show that $x = 0$ is globally uniformly asymptotically stable.

Proof. 1. Since (5a) and (5b), similar to Theorem 14.1, we can choose $r > 0$ and $\rho > 0$ such that $B_r \subset D$ and $\rho < \min_{\|x\| = r} W_1(x)$. Then, there exists an invariant set $\Omega_{t,\rho}$ such that

$$x_0 \in \{x \in B_r \mid W_2(x) \leq \rho\} \subset \Omega_{t,\rho} \quad \Rightarrow \quad \varphi(t) \in \Omega_{t,\rho}, \ \ \forall\, t \geq t_0 \geq 0.$$

According to (5c), for all $\forall\, t \geq t_0$, we have

$$V(t+\delta, \varphi(t+\delta)) \leq (1-\lambda) V(t, x).$$

Moreover, since (5.b), we have

$$V(s, \varphi(s)) \leq V(t, \varphi(t)), \ \ \forall\, s \in [t, t+\delta].$$

For any $t \geq t_0$, there exists the smallest positive integer N such that $t \leq t_0 + N\delta$. Divide the interval $[t_0, t_0 + (N-1)\delta]$ into $(N-1)$ equal subintervals of length δ each. Then,

$$V(t, \varphi(t)) \leq V(t_0 + (N-1)\delta, \varphi(t_0 + (N-1)\delta))$$

$$\leq (1-\lambda) V(t_0 + (N-2)\delta, \varphi(t_0 + (N-2)\delta))$$

$$\le \cdots \le (1-\lambda)^{(N-1)} V(t_0, x_0) = \frac{1}{1-\lambda}(1-\lambda)^N V(t_0, x_0)$$

$$\le \frac{1}{1-\lambda}(1-\lambda)^{\frac{t-t_0}{\delta}} V(t_0, x_0) .$$

Since $0 < 1-\lambda < 1$, let $b = \frac{1}{\delta}\ln\frac{1}{1-\lambda} > 0$, which implies that $0 < 1-\lambda = e^{-b\delta} < 1$. Then, we have

$$V(t, \varphi(t)) \le \frac{1}{1-\lambda}(1-\lambda)^{\frac{t-t_0}{\delta}} V(t_0, x_0) \le \frac{1}{1-\lambda} e^{-b(t-t_0)} V(t_0, x_0) .$$

Taking $\beta(r, s) = \frac{r}{1-\lambda} e^{-bs}$, which is a class KL function. The above inequality can be written as

$$V(t, \varphi(t)) \le \beta\,(V(t_0, x_0), t-t_0) .$$

Since $W_1(x)$ and $W_2(x)$ are positive definite, there exist α_1 and $\alpha_2 \in K$, defined on $[0, r]$, such that

$$W_1(x) \ge \alpha_1(\| x \|),\ W_2(x) \le \alpha_2(\| x \|) .$$

Therefore, any solution starting in $\Omega_{t,\rho}$ satisfies the inequality

$$\| \varphi(t) \| \le \alpha_1^{-1}(V(t, \varphi(t))) \le \alpha_1^{-1}(\beta(V(t_0, x_0), t-t_0))$$

$$\le \alpha_1^{-1}(\beta(\alpha_2(\| x_0 \|), t-t_0)) = \tilde{\beta}(\| x_0 \|, t-t_0)$$

for $x_0 \in \{x \in B_r \mid W_2(x) \le \rho\} \subset \Omega_{t,\rho}$. By Lemma 14.1 it shows that $\tilde{\beta}(\cdot\,,\cdot)$ is of class KL function. Thus, it implies that $x = 0$ of (TS) is UAS.

2. If $D = R^n$, $W_1(x)$ is radially unbounded, so is $W_2(x)$ by (14.5). Therefore, we can find $\alpha_2 \in K_\infty$ s.t. $W_2(x) \le \alpha_2(\| x \|)$. For any $x_0 \in R^n$, we choose $\rho > 0$ such that $\| x_0 \| \le \alpha_2^{-1}(\rho)$, i.e. $\alpha_2(\| x_0 \|) \le \rho$. Then, $W_2(x_0) \le \alpha_2(\| x_0 \|) \le \rho \Rightarrow x_0 \in \{x \in R^n \mid W_2(x) \le \rho\}$. The rest of the proof is the same as the above part for showing UAS. The end.